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ISRN INRIA/RR--8946--FR+ENG

ISSN 0249-6399

RESEARCH REPORT
N° 8946
August 2016
Project-Team Vegas

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Research Report n° 8946 — August 2016 — 13 pages

Abstract: Let X_n be a planar Poisson point process of intensity n. We give a new proof that the expected length of the Voronoi path between $(0,0)$ and $(1,0)$ in the Delaunay triangulation associated with X_n is $\frac{4}{\pi} \simeq 1.27$ when n goes to infinity; and we also prove that the variance of this length is $O(1/\sqrt{n})$. We investigate the length of possible shortcuts in this path, and defined a shortened Voronoi path whose expected length can be expressed as an integral that is numerically evaluated to $\simeq 1.16$. The shortened Voronoi path has the property to be *locally defined*; and is shorter than the previously known locally defined path in Delaunay triangulation such as the upper path whose expected length is $35/3\pi^2 \simeq 1.18$.

Key-words: Probabilistic analysis – Worst-case analysis – Walking algorithms

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Chemins dans la triangulation planaire de Poisson-Delaunay: Raccourcis dans la marche de Voronoi

Résumé : Soit X_n un processus ponctuel de Poisson planaire d'intensité n. Nous donnons une nouvelle démonstration que l'espérance de la longueur du chemin de Voronoï entre $(0,0)$ et $(1,0)$ dans la triangulation de Delaunay associée à X_n est $\frac{4}{\pi} \simeq 1.27$ quand n tends vers l'infini; nous démontrons aussi que la variance de cette longueur est $O(1/\sqrt{n})$. Nous étudions la longueurs gagnées par certains raccourcis dans le chemin de Voronoi et arrivons à exprimer cette longueur comme une intégrale dont l'évaluation numérique est $\simeq 1.16$. Le chemin de Voronoi raccourci a la propriété d'être *défini localement*; et il est plus court que les autres chemins défini localement déjà étudié tel que le *chemin supérieur* dont la longueur moyenne est $35/3\pi^2 \simeq 1.18$.

Mots-clés : Analyse probabiliste – Analyse dans le cas le pire – Algorithmes de marche

1 Introduction

The Delaunay triangulation is one of the most classical object of computational geometry and searching for paths in a point set using Delaunay edges is useful, e.g. for point location, nearest neighbor search [8], or routing in networks [3].

If the points are random, several walking strategies have been studied [2, 4, 6, 7, 9], in this paper we consider variations of a particular strategy called Voronoi path that consists in linking in order all the nearest neighbors of a point moving linearly from s to t where s and t are two points in the plane. We analyze these paths when s and t are two fixed points and when the point set is a Poisson point process of density n, possibly augmented by the two points s and t.

The Voronoi path is known to have an expected stretch factor $\frac{4}{\pi} \simeq 1.27$ when $n \to \infty$ [1], we provide an alternative proof of this result and prove that this length is quite stable by showing that the variance is small. Then we explore improvements on the Voronoi path by using some shortcuts. The length of one of this improved path can be expressed as an integral that we compute numerically getting an expected length of 1.16.

Any path in the Delaunay triangulation obviously yields an upper bound for the length of the shortest path. The best known upper bound being $\frac{35}{3\pi^2} \simeq 1.182$ which is obtained as the length of a path called *upper path* [6]. We say that a path is locally defined, if it can be decided if an edge belongs to the path between s and t by just knowing the neighborhood of the edge, s and t. Analyzing non locally defined paths, such as the shortest path is much more difficult than locally defined ones such as the upper path. Our improved Voronoi path is locally defined and gives a shorter alternative to the upper path.

2 Notations and Definitions

For a point set χ we define its Delaunay triangulation $\mathrm{Del}(\chi)$ as the set of edges $[p,q]$ with $p,q \in \chi$ such that there exist a disk D with $D \cap \chi = \{p,q\}$. One can remark that if there is such a disk, there is also such a disk so that p and q are on the boundary of the disk (shrink the first disk staying inside up to a position where the points are on the boundary).

The Voronoi diagram associated with χ is the tuple $(R_i)_{i\in\chi}$ where $\forall p \in \chi; R_p = \{q \in \mathbb{R}^2 / d(q,\chi) = d(q,p)\}$ (with d the Euclidean distance). R_p is the Voronoi cell of seed p.

The Voronoi Path $VP_\chi(s,t)$ between two points s and t is defined as the path formed by the seeds of the Voronoi cells intersecting the segment $\overline{st}$ (see Figure 1 for an example of Voronoi path). If $s,t \in \chi$ this path links s to t, otherwise it links the nearest neighbor of s to the nearest neighbor of t.

We denote $M(p_1,p_2)$ the intersection point between the bisector of p_1 and p_2 and the line (st). The ball centered at $M(p_1,p_2)$ passing through p_1 and p_2 is denoted $B(p_1,p_2)$ and its radius is denoted $R(p_1,p_2)$.

In the sequel our point set will be a Poisson point process X_n of intensity n or the same set augmented by two points $X = X_n \cup \{s,t\}$ where $s = (0,0)$ and $t = (1,0)$.

We denote $p_{i:j}$ the tuple of points $(p_i, p_{i+1}, \ldots, p_j)$, and $p_{i \neq j}$ the same tuple of points verifying $\forall k,l \in [i,j], p_k \neq p_l$.

3 Expectation of Stretch Factor of the Voronoi Path

The first lemma states that the fact that s and t belong to the point set has a small influence on the length of the Voronoi path when n is big:

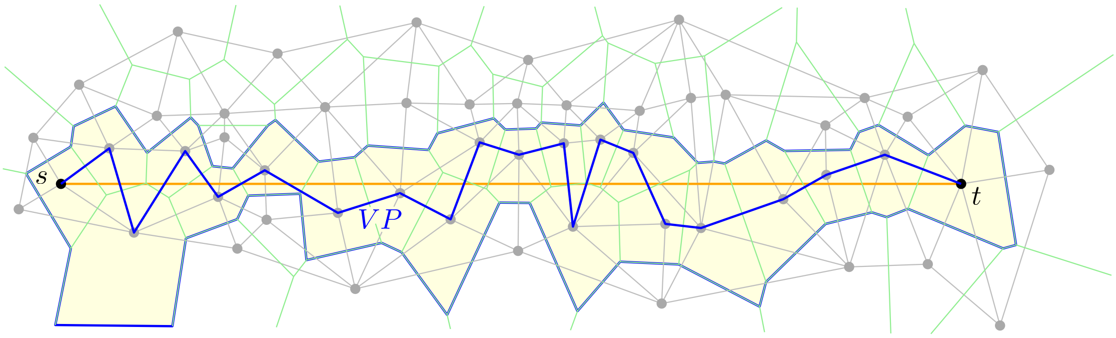


Figure 1: The Voronoi path

Lemma 1. *Let $X := X_n \cup \{s,t\}$ where X_n is a Planar poisson point process of intensity n and $s,t \in \mathbb{R}^2$. Let $\ell(VP_\chi(s,t))$ be the length of the Voronoi Path from s to t in $\mathrm{Del}(\chi)$. Then*

$$\mathbb{E}\left[\ell(VP_X(s,t))\right] = \mathbb{E}\left[\ell(VP_{X_n}(s,t))\right] + O\left(\frac{\|s-t\|}{\sqrt{n}}\right)$$

Proof. First, we remark that with very high probability $1-e^{-n\frac{\pi}{4}}$, the disk of diameter $[st]$ contains some points of X_n and thus any disk centered on $[st]$ of the form $B(\cdot,\cdot)$ cannot contain both s and t; we first assume this is the case, and no such ball does contain both s and t.

$VP_X(s,t)$ and $VP_{X_n}(s,t)$ only differ by few edges around s and t. Let sp_i be the first edge of $VP_X(s,t)$ and $p_1,p_2,\ldots,p_i$ be the i first vertices of $VP_{X_n}(s,t)$. First we remark that all p_j are neighbors of s in $\mathrm{Del}(X)$. actually, by definition of the paths, there is a disk D_i centered on a point in $[st]$ with p_i on its boundary, s inside and no points of X_n nor t inside, this disk witnesses that sp_i is a Delaunay edge in $\mathrm{Del}(X)$. Thus, to go from $VP_{X_n}(s,t)$ to $VP_X(s,t)$ we have to add one Delaunay edge incident to s: sp_i and to remove few edges between neighbors of s in $\mathrm{Del}(X)$. The length variation can be bounded using triangular inequality

$$\|p_1p_2\| + \|p_2p_3\| + \ldots + \|p_{i-1}p_i\| + \|sp_i\| \leq \|sp_1\| + 2\|sp_2\| + 2\|sp_3\| + \ldots + 2\|sp_{i-1}\| + 2\|sp_i\|$$

which is $O\left(\frac{\|s-t\|}{\sqrt{n}}\right)$ [6, Prop. 2.2]. The same applies to the end of the path around t.

In the rare case with an empty disk of diameter $[st]$ almost the same reasoning applies except that the two parts of $VP_{X_n}(s,t)$ to be removed may overlap. $O\left(\frac{\|s-t\|}{\sqrt{n}}\right)$ is still an upper bound on the length of the removed part. Now the added part is just edge st of length one, but since it arises only with probability $e^{-n\frac{\pi}{4}} = o\left(\frac{\|s-t\|}{\sqrt{n}}\right)$ the result still holds. □

Theorem 2. *X_n is a Planar poisson point process of intensity n and $s,t \in \mathbb{R}^2$. Let $\ell(VP_{X_n}(s,t))$ be the length of the Voronoi Path from s to t in $\mathrm{Del}(X_n)$. Then*

$$\mathbb{E}\left[\frac{\ell(VP_{X_n}(s,t))}{\|s-t\|}\right] = \frac{4}{\pi}.$$

Proof. Without loss of generality, we may assume that $s,t = (0,0),(1,0)$

$$\ell(VP_{X_n}(s,t)) = \frac{1}{2}\sum_{p_1 \neq p_2 \in X_n^2} \mathbb{1}_{[\overline{p_1p_2} \in VP_{X_n}(s,t)]} ||p_2 - p_1||,$$

$$\ell(VP_{X_n}(s,t)) = \tfrac{1}{2} \sum_{p_1 \neq_{\cdot 2} \in X_n^2} \mathbb{1}_{[M(p_1,p_2)\in\overline{st}]} \mathbb{1}_{[B(p1,p2)\cap X_n=\varnothing]} ||p_2 - p_1||,$$

Using Slivnyak-Mecke formula, we transform this sum in an integral [10, Theorem 3.3.5]:

$$\mathbb{E}\left[\ell(VP_{X_n}(s,t))\right] = \tfrac{n^2}{2} \int_{(\mathbb{R}^2)^2} \mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\, \mathbb{P}\left[B(p1,p2)\cap X_n = \varnothing\right] ||p_2 - p_1|| dp_{1:2}.$$

Let Φ be the function

$$\begin{array}{llcl} \Phi: & \mathbb{R}\times\mathbb{R}_+\times[0,2\pi)^2 & \longrightarrow & \mathbb{R}^2\times\mathbb{R}^2 \\ & (x,r,\alpha_1,\alpha_2) & \longmapsto & (p_1,p_2), \end{array}$$

where for i=1,2 we let

$$p_i = (x,0) + r(\cos\alpha_i, \sin\alpha_i).$$

As long as p_1 and p_2 do not have the same absciss, which occurs with probability 1, x is the absciss of $M(p_1,p_2)$. r is the distance between this point and p_1. So Φ is a C^1-diffeomorphism up to a null set. Its Jacobian is

$$\det(J_\Phi) = \begin{vmatrix} 1 & \cos\alpha_1 & -r\sin\alpha_1 & 0 \\ 0 & \sin\alpha_1 & r\cos\alpha_1 & 0 \\ 1 & \cos\alpha_2 & 0 & -r\sin\alpha_2 \\ 0 & \sin\alpha_2 & 0 & r\cos\alpha_2 \end{vmatrix} = r^2(\cos\alpha_2 - \cos\alpha_1).$$

Since $\|p_2 - p_1\| = 2r\left|\sin\frac{\alpha_1-\alpha_2}{2}\right|$ and $\mathbb{P}\left[B(0,r)\cap X_n = \varnothing\right] = e^{-n\pi r^2}$, we get

$$\begin{aligned}
&\mathbb{E}\left[\ell(VP_{X_n}(s,t))\right] \\
&\quad = \tfrac{n^2}{2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{[0,2\pi)^2} \mathbb{1}_{[0<x<1]} e^{-n\pi r^2} 2r \left|\sin\frac{\alpha_1-\alpha_2}{2}\right| |\det(J_\Phi)|\, d\alpha_1 d\alpha_2 dr dx \\
&= n^2 \int_{-\infty}^{\infty} \mathbb{1}_{[0<x<1]} dx \int_0^{\infty} e^{-n\pi r^2} r^3 dr \int_{[0,2\pi)^2} \left|\sin\frac{\alpha_1-\alpha_2}{2}\right| |\cos\alpha_2 - \cos\alpha_1|\, d\alpha_1 d\alpha_2 \\
&= n^2 \frac{1}{2\pi^2 n^2} \int_{[0,2\pi)^2} \left|\sin\frac{\alpha_1-\alpha_2}{2}\right| |\cos\alpha_2 - \cos\alpha_1|\, d\alpha_1 d\alpha_2 \\
&= n^2 \frac{1}{2\pi^2 n^2} \times 8\pi = \frac{4}{\pi}.
\end{aligned}$$

□

The above trigonometric integral is invariant by substituting (α_2,α_1) or $(2\pi-\alpha_1, 2\pi-\alpha_2)$ to (α_1,α_2). Thus:

$$\begin{aligned}
&\int_{[0,2\pi)^2} \left|\sin\frac{\alpha_1-\alpha_2}{2}\right| |\cos\alpha_2 - \cos\alpha_1|\, d\alpha_1 d\alpha_2 \\
&\qquad = 4\int_0^{\pi}\int_{\alpha_2}^{2\pi-\alpha_2} \sin\frac{\alpha_1-\alpha_2}{2} (\cos\alpha_2 - \cos\alpha_1)\, d\alpha_1 d\alpha_2 = 8\pi.
\end{aligned}$$

4 Variance of Stretch Factor of the Voronoi Path

Theorem 3.

$$\mathbb{V}\left[\frac{\ell(VP_{X_n})}{\|s-t\|}\right] = O(n^{-\frac{1}{2}}).$$

Proof. Once again, we assume that $s=(0,0)$ and $t=(1,0)$.

$$\mathbb{V}\left[\ell(VP_{X_n})\right] = \mathbb{E}\left[\ell(VP_{X_n})^2\right] - \mathbb{E}\left[\ell(VP_{X_n})\right]^2,$$

$$\ell(VP_{X_n})^2 = \left(\tfrac{1}{2}\sum_{p_{1\neq..2}\in {X_n}^2} \ell_{VP}(p_{1:2})\right)^2,$$

where $\ell_{VP}(p_{1:2}) = \mathbb{1}_{[B(p_1,p_2)\cap X_n=\varnothing]}\mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\|p1-p2\|$

$$\begin{aligned}\ell(VP_{X_n})^2 = \tfrac{1}{4}&\left(2\sum_{p_{1\neq..2}\in {X_n}^2} \ell_{VP}(p_{1:2})^2\right)\\ &+\tfrac{1}{4}\left(4\sum_{p_{1\neq..3}\in {X_n}^3} \ell_{VP}(p_{1:2})\ell_{VP}(p_{2:3}))\right)\\ &+\tfrac{1}{4}\left(\sum_{p_{1\neq..4}\in {X_n}^4} \ell_{VP}(p_{1:2})\ell_{VP}(p_{3:4}))\right).\end{aligned}$$

$$\mathbb{E}\left[\left(\sum_{p_{1\neq..4}\in {X_n}^4} \ell_{VP_{X_n}}(p_{1:2})\ell_{VP_{X_n}}(p_{3:4}))\right)\right] = n^4\int_{(\mathbb{R}^2)^4}\mathbb{E}\left[\ell_{VP_{X'}}(p_{1:2})\ell_{VP_{X'}}(p_{3:4})\right]dp_{1:4},$$

where $X' = X_n\cup\{p_{1:4}\}$

$$\begin{aligned}&\mathbb{E}\left[\left(\sum_{p_{1\neq..4}\in {X_n}^4} \ell_{VP_{X_n}}(p_{1:2})\ell_{VP_{X_n}}(p_{3:4}))\right)\right]\\ &\quad= n^4\int_{(\mathbb{R}^2)^4}\mathbb{E}\left[\mathbb{1}_{[B(p_1,p_2)\cap X'=\varnothing]}\mathbb{1}_{[B(p_3,p_4)\cap X'=\varnothing]}\right]\mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\\ &\qquad\qquad \mathbb{1}_{[M(p_3,p_4)\in\overline{st}]}\|p_1-p_2\|\|p_3-p_4\|dp_{1:4}\\ &\quad\leq n^4\int_{(\mathbb{R}^2)^4}\mathbb{E}\left[\mathbb{1}_{[B(p_1,p_2)\cap X_n=\varnothing]}\mathbb{1}_{[B(p_3,p_4)\cap X_n=\varnothing]}\right]\mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\\ &\qquad\qquad \mathbb{1}_{[M(p_3,p_4)\in\overline{st}]}\|p_1-p_2\|\|p_3-p_4\|dp_{1:4}.\end{aligned}$$

With the same substitution as previously, done twice, we get:

$$\mathbb{E}\left[\left(\sum_{p_{1 \neq .4} \in X_n{}^4} \ell_{VP_{X_n}}(p_{1:2})\ell_{VP_{X_n}}(p_{3:4})\right)\right]$$
$$\leq n^4 \int_{[0,1]^2}\int_{\mathbb{R}_+^2}\int_{[0,2\pi)^4} e^{-n\,\mathcal{A}\left(B((x,0),r)\cup B((x',0),r')\right)} 2r\left|\sin\frac{\alpha_1-\alpha_2}{2}\right| 2r'\left|\sin\frac{\alpha_3-\alpha_4}{2}\right|$$
$$r^2|\cos\alpha_2-\cos\alpha_1|r'^2|\cos\alpha_4-\cos\alpha_3|d\alpha_{1:4}drdr'dxdx'.$$

By rewriting the exponential as:
$e^{-n\,\mathcal{A}\left(B((x,0),r)\cup B((x',0),r')\right)} = e^{-n\pi(r^2+r'^2)} + \left(e^{-n\,\mathcal{A}\left(B((x,0),r)\cup B((x',0),r')\right)} - e^{n\pi(r^2+r'^2)}\right)$
and applying Fubini's theorem, we get:

$$\mathbb{E}\left[\frac{1}{4}\left(\sum_{p_{1 \neq .4} \in X_n{}^4} \ell_{VP_{X_n}}(p_{1:2})\ell_{VP_{X_n}}(p_{3:4})\right)\right] \leq \mathbb{E}\left[\ell(VP_{X_n})\right]^2 + r_n,$$

where

$$r_n = n^4 \int_{[0,1]^2}\int_{\mathbb{R}_+^2}\int_{[0,2\pi)^4} \left(e^{-n\,\mathcal{A}\left(B((x,0),r)\cup B((x',0),r')\right)} - e^{-n\pi(r^2+r'^2)}\right) 2r\left|\sin\frac{\alpha_1-\alpha_2}{2}\right|$$
$$2r'\left|\sin\frac{\alpha_3-\alpha_4}{2}\right| r^2|\cos\alpha_2-\cos\alpha_1|r'^2|\cos\alpha_4-\cos\alpha_3|d\alpha_{1:4}drdr'dxdx'.$$

Breaking the symmetry between r and r', we get

$$r_n = 8n^4 \int_{[0,1]^2}\int_{\mathbb{R}_+}\int_{r'}^{\infty} \left(e^{-n\,\mathcal{A}\left(B((x,0),r)\cup B((x',0),r')\right)} - e^{-n\pi(r^2+r'^2)}\right) r^3r'^3drdr'dxdx'$$
$$\times \int_{[0,2\pi)^4}\left|\sin\frac{\alpha_1-\alpha_2}{2}\sin\frac{\alpha_3-\alpha_4}{2}(\cos\alpha_2-\cos\alpha_1)(\cos\alpha_4-\cos\alpha_3)\right| d\alpha_{1:4}.$$

Since we now have $r' \leq r$, we get

$$\left(e^{-n\,\mathcal{A}\left(B((x,0),r)\cup B((x',0),r')\right)} - e^{-n\pi(r^2+r'^2)}\right) \leq e^{-n\pi r^2},$$

$$\begin{aligned}
r_n &\leq 8(8\pi)^2 n^4 \int_{[0,1]^2}\int_0^\infty\int_0^r r^6 e^{-n\pi r^2}\mathbb{1}_{[B(z,r)\cap B(z',r')\neq\varnothing]}dr'drdxdx' \\
&\leq 512\pi^2 n^4 \int_0^1\int_{x'-2r}^{x'+2r}\int_0^\infty r^7 e^{-n\pi r^2}drdxdx' \\
&\leq 512\pi^2 n^4 \int_0^\infty 4r^8 e^{-n\pi r^2}dr \\
&\leq 512\pi^2 n^4 \cdot 4\frac{105}{32\pi^4 n^4\sqrt{n}} = O(n^{-\frac{1}{2}}).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\sum_{p_{1\neq .2}\in X_n{}^2} \ell_{VP_{X_n}}(p_{1:2})^2\right]\\
&= n^2\int_{(\mathbb{R}^2)^2}\mathbb{E}\left[\mathbb{1}_{[B(p_1,p_2)\cap X_n=\varnothing]}\mathbb{1}_{[M(p_1,p_2\in\overline{st})]}\|p_1-p_2\|^2\right]dp_1dp_2\\
&= n^2\int_0^1\int_0^\infty\int_{[0,2\pi)^2} e^{-n\pi r^2}4r^2\sin^2\frac{\alpha_1-\alpha_2}{2}r^2|\cos\alpha_1-\cos\alpha_2|d\alpha_1d\alpha_2drdx\\
&= 4n^2\int_0^1 dx\int_0^\infty e^{-n\pi r^2}r^4dr\int_{[0,2\pi)^2}\sin^2\frac{\alpha_1-\alpha_2}{2}|\cos\alpha_1-\cos\alpha_2|d\alpha_1d\alpha_2\\
&= 4n^2\frac{3}{4\pi n^{\frac52}}\frac{16}{3}=O(n^{-\frac12}).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\sum_{p_{1\neq .3}\in X_n{}^3}\ell_{VP_{X_n}}(p_{1:2})\ell_{VP_{X_n}}(p_{2:3})\right]\\
&\le n^3\int_{(\mathbb{R}^2)^3}\mathbb{E}\left[\mathbb{1}_{[B(p_1,p_2)\cap X_n=\varnothing]}\mathbb{1}_{[B(p_2,p_3)\cap X_n=\varnothing]}\mathbb{1}_{[M(p_1,p_2\in\overline{st})]}\mathbb{1}_{[M(p_2,p_3\in\overline{st})]}\|p_1-p_2\|\|p_2-p_3\|\right]\\
&\qquad dp_1dp_2dp_3\\
&\le 2n^3\int_{(\mathbb{R}^2)^3}e^{-n\pi R(p_1,p_2)^2}\mathbb{1}_{[\|p_2-p_3\|\le\|p_1-p_2\|]}\mathbb{1}_{[M(p_1,p_2\in\overline{st})]}\|p_1-p_2\|\|p_2-p_3\|dp_1dp_2dp_3\\
&\le 2n^3\int_{(\mathbb{R}^2)^3}e^{-n\pi R(p_1,p_2)^2}\mathbb{1}_{[p_3\in B(M(p_1,p_2),3R(p_1,p_2))]}\mathbb{1}_{[M(p_1,p_2\in\overline{st})]}4R(p_1,p_2)^2dp_1dp_2dp_3\\
&= 8n^3\int_{(\mathbb{R}^2)^2}e^{-n\pi R(p_1,p_2)^2}9\pi R(p_1,p_2)^2\mathbb{1}_{[M(p_1,p_2\in\overline{st})]}R(p_1,p_2)^2dp_1dp_2\\
&= 72\pi n^3\int_0^1 dx\int_0^\infty e^{-n\pi r^2}r^2r^2r^2dr\int_{[0,2\pi)^2}|\cos\alpha_2-\cos\alpha_1|d\alpha_2d\alpha_1\\
&\le 72\pi n^3\cdot\frac{15}{16\pi^3n^{\frac72}}\cdot 8=O(n^{-\frac12}).
\end{aligned}$$

Combining these terms in the definition of $\mathbb{V}\left[\ell(VP_{X_n})\right]$ terminates the proof. □

5 Improvement upon the Voronoi Path

We call shortcut of VP_χ a triangle (p_1,p_2,p_3) such that (p_1,p_2) and (p_2,p_3) are in the Voronoi Path, and (p_1,p_2,p_3) is in Del(χ) (see Figure 2).

Notice that it may exist other shortcuts replacing more than two edges in the Voronoi path, but the probability of existence decrease with the length of the replaced chain. In this paper we limit our interest to the above defined simple shortcuts.

Let $\ell_{SC}(p_1,p_2,p_3)$ be defined as the length saved by taking the shortcut (p_1,p_2,p_3), i.e $\ell_{SC}(p_1,p_2,p_3)=\|p_1-p_2\|+\|p_2-p_3\|-\|p_1-p_3\|$.

As shown on Figure 2 some shortcuts are incompatible, but the set of shortcuts can be divided in two sets, the shortcuts above the Voronoi path and the shortcuts below the Voronoi path that

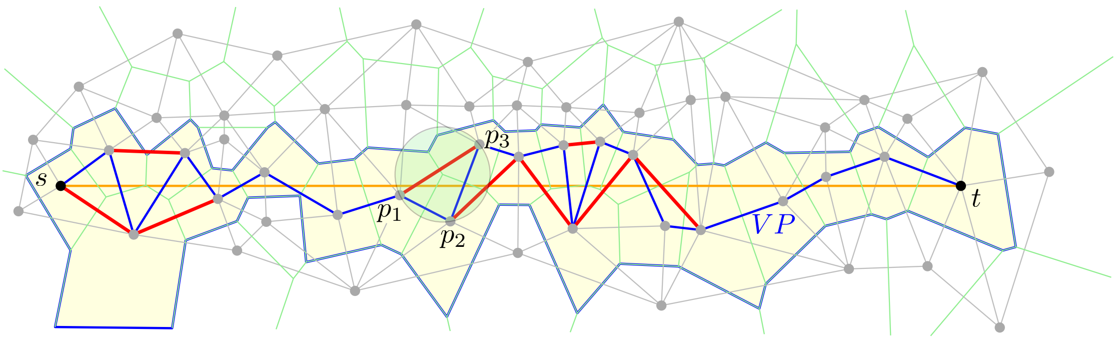


Figure 2: Shortcuts in Voronoi path of Figure 1 (in red).

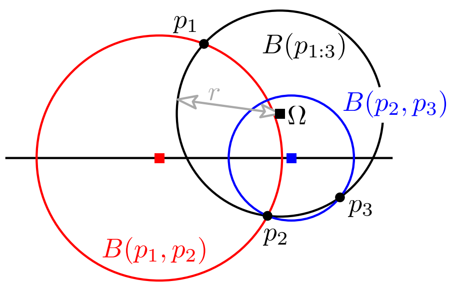
are compatible. By symmetry, the expected length of the above shortcuts is equal to the one of below shortcuts and is equal to half the total length of all shortcuts. Let $gain_{X_n}$ denote this expected saving in the Voronoi path for a Poisson point process X_n.

$$\mathbb{E}\left[gain_{X_n}\right] = \mathbb{E}\left[\frac{1}{2}\sum_{p_{1\neq .3}\in X_n{}^3} \mathbb{1}_{[p_{1:3}\in \mathrm{Del}(X_n)]}\mathbb{1}_{[p_{1:2}\in VP_{X_n}]}\mathbb{1}_{[p_{2:3}\in VP_{X_n}]}\mathbb{1}_{[x_{p_1}<x_{p_2}<x_{p_3}]}\ell_{SC}(p_{1:3})\right]$$

.

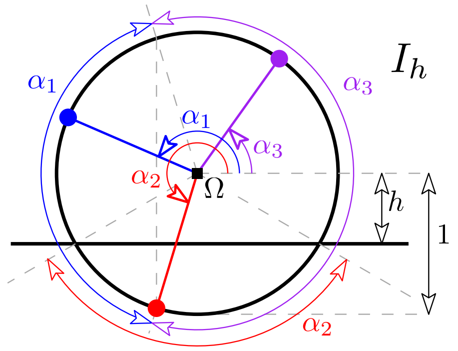
We use the Slivnyak-Mecke formula:

$$\begin{aligned}
\mathbb{E}\left[gain_{X_n}\right] &= \frac{n^3}{2}\int_{(\mathbb{R}^2)^3} \mathbb{E}\Big[\mathbb{1}_{[p_{1:3}\in \mathrm{Del}(X_n)]}\mathbb{1}_{[p_{1:2}\in VP_{X_n\cup\{p_3\}}]}\mathbb{1}_{[p_{2:3}\in VP_{X_n\cup\{p_1\}}]} \\
&\qquad\qquad \mathbb{1}_{[x_{p_1}<x_{p_2}<x_{p_3}]}\ell_{SC}(p_{1:3})\Big]\, dp_{1:3} \\
&= \frac{n^3}{2}\int_{(\mathbb{R}^2)^3} \mathbb{P}\left[(B(p_{1:3})\cup B(p_1,p_2)\cup B(p_2,p_3))\cap X_n = \varnothing\right]\mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\mathbb{1}_{[M(p_2,p_3)\in\overline{st}]} \\
&\qquad\qquad \mathbb{1}_{[p_1\notin B(p_2,p_3)]}\mathbb{1}_{[p_3\notin B(p_1,p_2)]}\mathbb{1}_{[x_{p_1}<x_{p_2}<x_{p_3}]}\ell_{SC}(p_{1:3})dp_{1:3} \\
&= n^3\int_{(\mathbb{R}^2)^3} e^{-n\,\mathcal{A}(B(p_{1:3})\cup B(p_1,p_2)\cup B(p_2,p_3))}\mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\mathbb{1}_{[M(p_2,p_3)\in\overline{st}]}\mathbb{1}_{[y_{p_2}<0]} \\
&\qquad\qquad \mathbb{1}_{[p_1\notin B(p_2,p_3)]}\mathbb{1}_{[p_3\notin B(p_1,p_2)]}\mathbb{1}_{[x_{p_1}<x_{p_2}<x_{p_3}]}\ell_{SC}(p_{1:3})dp_{1:3}.
\end{aligned}$$



We have limited our attention to half of the shortcuts using the assumption that $y_{p_2} < 0$. The assumption $x_{p_1} < x_{p_2} < x_{p_3}$ ensure that each triangle is counted only once. Under these two hypotheses, we consider only a subset of the possible shortcuts that verify $y_\Omega > 0$ (with Ω the center of $B(p_{1:3})$) and p_1, p_2, p_3 counterclockwise (ccw) then $p_1 \notin B(p_2,p_3)$ and $p_3 \notin B(p_1,p_2)$. Since we do not consider all shortcuts any longer, in the sequel, we only have an upper bound on $gain_{X_n}$. Actually our experiments show that this is a reasonably good upper bound.

$$\mathbb{E}\left[gain_{X_n}\right] \geq n^3 \int_{(\mathbb{R}^2)^3} e^{-n\,\mathcal{A}(B(p_{1:3})\cup B(p_1,p_2)\cup B(p_2,p_3))} \mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\mathbb{1}_{[M(p_2,p_3)\in\overline{st}]}\mathbb{1}_{[y_{p_2}<0]}$$
$$\mathbb{1}_{[y_\Omega>0]}\mathbb{1}_{[p_1,p_2,p_3\ \text{ ccw}]}\mathbb{1}_{[x_{p_1}<x_{p_2}<x_{p_3}]}\ell_{SC}(p_{1:3})dp_{1:3} = E_1.$$



Let r be the radius of $B(p_{1:3})$,

$$S(\tfrac{y_\Omega}{r},\alpha_1,\alpha_2,\alpha_3) = \frac{\mathcal{A}(B(p_{1:3})\cup B(p_1,p_2)\cup B(p_2,p_3))}{r^2}$$

be the area of the union of the three balls normalized by r^2, and $h = \frac{y_\Omega}{r}$ the normalized distance from Ω to line (st). Since Ω is assumed above line (st) and p_2 below it, $h \in [0,1]$. We define a region for the angles α_i:

$$I_h = \left\{ (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3 \begin{cases} 2\pi-\alpha_2 < \alpha_1 < \alpha_2 \\ \pi + \arcsin h < \alpha_2 < 2\pi - \arcsin h \\ \alpha_2 - 2\pi < \alpha_3 < 2\pi - \alpha_2 \end{cases} \right\}.$$

E_1 becomes:

$$E_1 = n^3 \int_{(\mathbb{R}^2)^3} e^{-nr^2 S(\frac{y_\Omega}{r},\alpha_1,\alpha_2,\alpha_3)}\mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\mathbb{1}_{[M(p_2,p_3)\in\overline{st}]}\mathbb{1}_{[0<y_\Omega<r]}\mathbb{1}_{[\alpha_{1:3}\in I_h]}\ell_{SC}(p_{1:3})dp_{1:3}$$

Basic trigonometry gives: $x_{M(p_1,p_2)} = x_\Omega - \frac{y_\Omega}{\tan(\frac{\alpha_1+\alpha_2}{2})}$ and $x_{M(p_2,p_3)} = x_\Omega - \frac{y_\Omega}{\tan(\frac{\alpha_2+\alpha_3}{2})}$. Defining

$$J_{y_\Omega,\alpha_{1:3}} = \begin{cases} \left[\frac{y_\Omega}{\tan(\frac{\alpha_1+\alpha_2}{2})}, 1+\frac{y_\Omega}{\tan(\frac{\alpha_2+\alpha_3}{2})}\right] & \text{, if } \frac{y_\Omega}{\tan(\frac{\alpha_1+\alpha_2}{2})} \leq 1 + \frac{y_\Omega}{\tan(\frac{\alpha_2+\alpha_3}{2})} \\ \varnothing & \text{, otherwise} \end{cases},$$

we have $\mathbb{1}_{[M(p_1,p_2)\in\overline{st}]}\mathbb{1}_{[M(p_2,p_3)\in\overline{st}]} = \mathbb{1}_{[x_\Omega\in J_{y_\Omega,\alpha_{1:3}}]}$. We are now ready to substitute the variables using Blaschke-Petkantschin formula [10, Theorem 7.3.1]. We get:

$$\begin{aligned} E_1 &= n^3 \int_{[0,2\pi)^3}\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-nr^2S(\frac{y_\Omega}{r},\alpha_1,\alpha_2,\alpha_3)}\mathbb{1}_{[x_\Omega\in J_{y_\Omega,\alpha_{1:3}}]}\mathbb{1}_{[0<y_\Omega<r]}\mathbb{1}_{[\alpha_{1:3}\in I_h]} \\ &\qquad 2r\left(\left|\sin\tfrac{\alpha_1-\alpha_2}{2}\right| + \left|\sin\tfrac{\alpha_2-\alpha_3}{2}\right| - \left|\sin\tfrac{\alpha_1-\alpha_3}{2}\right|\right)2r^3\,\mathcal{A}\left(\alpha_{1:3}\right)dy_\Omega dx_\Omega dr d\alpha_{1:3}, \\ &= 2n^3\int_{[0,2\pi)^3}\int_0^\infty\int_0^r\left(\int_{J_{y,\alpha_{1:3}}}dx\right)e^{-nr^2S(\frac{y}{r},\alpha_1,\alpha_2,\alpha_3)}\mathbb{1}_{[\alpha_{1:3}\in I_h]} \\ &\qquad g(\alpha_{1:3})r^4 dy dr d\alpha_{1:3}, \end{aligned}$$

where $g(\alpha_{1:3}) = 2\,\mathcal{A}\left(\alpha_{1:3}\right)\left(\left|\sin\frac{\alpha_1-\alpha_2}{2}\right| + \left|\sin\frac{\alpha_2-\alpha_3}{2}\right| - \left|\sin\frac{\alpha_1-\alpha_3}{2}\right|\right)$,

$$E_1 \geq 2n^3\int_{[0,2\pi)^3}\int_0^1\int_0^\infty\left(1-rh\ g'(\alpha_{1:3})\right)e^{-nr^2S(h,\alpha_1,\alpha_2,\alpha_3)}r^5\mathbb{1}_{[\alpha_{1:3}\in I_h]}g(\alpha_{1:3})drdhd\alpha_{1:3},$$

where $g'(\alpha_{1:3}) = \frac{1}{\tan(\frac{\alpha_1+\alpha_2}{2})} - \frac{1}{\tan(\frac{\alpha_2+\alpha_3}{2})}$. The length of $J_{y,\alpha_{1:3}}$ is $1 - rh\ g'(\alpha_{1:3})$ when the interval is nonempty; otherwise, $1 - rh\, g'(\alpha_{1:3})$ is negative and still bounds the interval's length from below.

Since $n^3 \int_0^\infty e^{-nr^2 S(h,\alpha_1,\alpha_2,\alpha_3)} r^6 dr = O\left(n^{-\frac{1}{2}}\right)$ the contribution of the term rhg' to E_1 is negligible and we get

$$\begin{aligned}
E_1 &\geq 2n^3 \int_{[0,2\pi)^3} \int_0^1 \left(\int_0^\infty e^{-nr^2 S(h,\alpha_1,\alpha_2,\alpha_3)} r^5 dr\right) \mathbb{1}_{[\alpha_{1:3}\in I_h]} g(\alpha_{1:3}) dh d\alpha_{1:3} + O\left(n^{-\frac{1}{2}}\right) \\
&= 2n^3 \int_{[0,2\pi)^3} \int_0^1 \left(\frac{1}{n^3 S(h,\alpha_1,\alpha_2,\alpha_3)^3}\right) \mathbb{1}_{[\alpha_{1:3}\in I_h]} g(\alpha_{1:3}) dh d\alpha_{1:3} + O\left(n^{-\frac{1}{2}}\right) \\
&= 2\int_0^1 \int_{I_h} \frac{g(\alpha_{1:3})}{S(h,\alpha_1,\alpha_2,\alpha_3)^3} dh d\alpha_{1:3} + O\left(n^{-\frac{1}{2}}\right). \qquad (1)
\end{aligned}$$

Determination of $S(h, \alpha_1, \alpha_2, \alpha_3)$

$$S(h,\alpha_1,\alpha_2,\alpha_3) = \mathcal{A}\left(B(u_{1:3}) \cup B(u_1,u_2) \cup B(u_2,u_3)\right)$$

with $u_i = (\cos\alpha_i, \sin\alpha_i)$.

$$\begin{aligned}
S(h,\alpha_1,\alpha_2,\alpha_3) = {} & \mathcal{A}\left(B(u_{1:3})\right) + \mathcal{A}\left(B(u_1,u_2)\right) + \mathcal{A}\left(B(u_2,u_3)\right) \\
& - \mathcal{A}\left(B(u_{1:3}) \cap B(u_1,u_2)\right) - \mathcal{A}\left(B(u_{1:3}) \cap B(u_2,u_3)\right) \\
& - \mathcal{A}\left(B(u_1,u_2) \cap B(u_2,u_3)\right) + \mathcal{A}\left(B(u_{1:3}) \cap B(u_1,u_2) \cap B(u_2,u_3)\right).
\end{aligned}$$

We will look at the different terms of this sum. First we remark that when $\alpha_{1:3} \in I_h$, the two last terms disappear since $B(u_1,u_2) \cap B(u_2,u_3) \subset B(u_{1:3})$ (the apexes of $B(u_1,u_2) \cap B(u_2,u_3)$ are p_2 and its symetric with respect to the line $y = -h$).

The first term is just π the area of the unit circle.

The second and third terms are πr_{12}^2 and πr_{23}^2 with $r_{12} = R(u_1,u_2)$ and $r_{23} = R(u_2,u_3)$.

The fourth term is

$$\mathcal{A}\left(B(u_{1:3}) \cap B(u_1,u_2)\right) = \tfrac{1}{2} r_{12}^2(\theta_{12} - |\sin\theta_{12}|) + \tfrac{1}{2}(\phi_{12} - |\sin\phi_{12}|),$$

with $\phi_{12} = \widehat{p_1 \Omega p_2}$ and $\theta_{12} = \widehat{p_2 M(p_1,p_2) p_1}$

The fifth term is similar to the fourth one.

Above undefined quantites can be expressed in term of h and $\alpha_{1:3}$. Since the angle of $\Omega M(u_1,u_2)$ is $\frac{\pi}{2} - \frac{\alpha_1+\alpha_2}{2}$ and $y_{M(u_1,u_2)} = -h$ we have $x_{M(u_1,u_2)} = -h\tan\frac{\pi+\alpha_1+\alpha_2}{2}$. We deduce $r_{12}^2 = (sin\alpha_2 + h)^2 + (\cos\alpha_2 + h\tan\frac{\pi+\alpha_1+\alpha_2}{2})^2$.

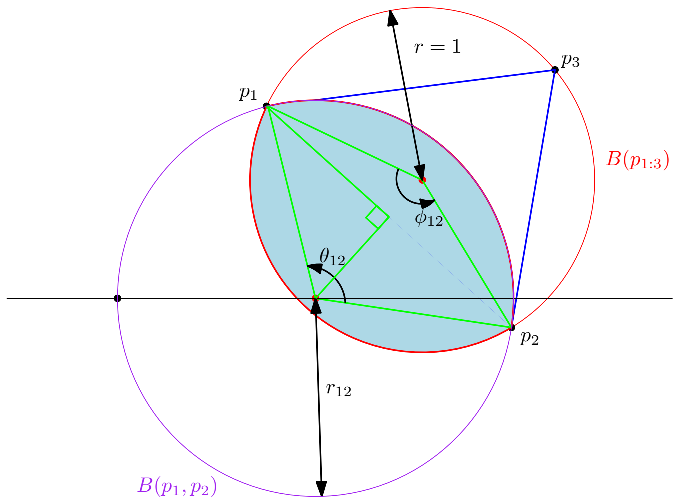
Similarly $r_{23}^2 = (sin\alpha_2 + h)^2 + (\cos\alpha_2 + h\tan\frac{\pi+\alpha_3+\alpha_2}{2})^2$.

The angles ϕ are easy to compute: $\phi_{12} = \alpha_2 - \alpha_1$ and $\phi_{23} = \alpha_3 - \alpha_2 + 2\pi$.

The angle θ_{12} verifies $\left|\cos\frac{\theta_{12}}{2}\right| = \frac{\|M(u_1,u_2)\frac{u_1+u_2}{2}\|}{r_{12}} = \frac{\sqrt{r_{12}^2 - \frac{\|u_1u_2\|}{2}}}{r_{12}} = \sqrt{1 - \left(\frac{\sin\frac{\alpha_1-\alpha_2}{2}}{r_{12}}\right)^2}$,

$\cos\frac{\theta_{12}}{2} \geq 0$ iff $0 \leq 2h + \sin\alpha_1 + \sin\alpha_2$, thus

$$\theta_{12} = 2\arccos\left(\sqrt{1 - \left(\frac{\sin\frac{\alpha_1-\alpha_2}{2}}{r_{12}}\right)^2}\ \mathrm{sign}(2h + \sin\alpha_1 + \sin\alpha_2)\right).$$



By a very similar reasoning we get

$$\theta_{23} = 2\arccos\left(\sqrt{1-\left(\frac{\sin\frac{\alpha_2-\alpha_3}{2}}{r_{23}}\right)^2}\,\mathrm{sign}(2h+\sin\alpha_2+\sin\alpha_3)\right).$$

Value of the gain

Using the above expression for $S(h,\alpha_{1:3})$, the integral of Equation (1) has been numerically approximated using Maple giving $\mathbb{E}\left[gain_{X_n}\right] \simeq 0.108$. So the expectation of the length of the new path is 1.165.

Maple file is available with this research report.

6 Alternative Paths

As a concluding remark, we mention several possibilities of paths from s to t that can be defined in a Delaunay triangulation:
- the shortest path,
- compass routing (vertex following v minimizing the angle with vt),
- upper path (edges vw of triangles uvw with u below line (st) and v and w above),
- greedy angle (vertex following v minimizing the angle with the horizontal through v admist the vertices of edges intersecting line (st)),
- closest neighbor (vertex following v minimizing the distance to t),
- the Voronoi path,
- the Voronoi path with all possible shortcuts taken greedily, and
- the Voronoi path with half the shortcuts (i.e. ccw shortcuts).

Some of these paths are locally defined. i.e., the fact that vw belongs to the path can be decided knowing only s, t and some neighborhood of vw. Some are incremental, i.e., the vertex following v can be decided knowing that v is on the path, s, t, and some neighborhood of v.

Using CGAL [5] we experiment on the length of these paths (with $\|st\|=1$) for a random set of points. The length of number of edges that we obtained after 1000 experiments with point density 10^6 are in the following table:

Path	Experimental Length	Number of edges	Path properties	Theoretical bound
Shortest path	1.041	927		$\in [1+10^{-11}, 1.182]$ [6]
Compass routing	1.068	956	incremental	$\Theta(\sqrt{n})$ edges [7]
Greedy angle	1.097	997	incremental	
VP greedy shortcuts	1.130	995	incremental	
VP ccw shortcuts	1.164	1081	incremental locally defined	1.165 [this paper, numerical integration]
Closest neighbor walk	1.167	873	incremental	$\Theta(\sqrt{n})$ edges [7]
Upper path	1.177	1072	locally defined	$\frac{35}{3\pi^2} \simeq 1.182$ [6]
VP	1.274	1273	incremental locally defined	$\frac{4}{\pi} \simeq 1.273$ [1]

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Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
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ISSN 0249-6399